An Extended Frame for Thevenin (Norton) Theorem

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Abstract: The Helmholtz-Thevenin-Norton theorem is widely used in circuit calculations. However, it also is an important "junction" of some topics deserving heuristic (re)consideration. The need for such a discussion arose because of a recent focusing [1] on a specific circuit topology that is usually (and rather unjustly) ignored. It is stressed that if the equivalent active 1-port is a load of an external active circuit, then the latter circuit sees the 1-port as a nonlinear resistor. This observation can be associated with the necessity of detailed discussion of nonlinear resistors already at an early educational stage, which agrees with the position of *Desoer&Kuh*, ignored in many standard textbooks.

An open question

To what degree we have to, and are allowed to, rework the basic topics, or foundations of a basic discipline?

Being concerned, in particular, with the pedagogical aspect, we can try to make this not easy question more feasible, by using, for a moment, the informal analogy between the academic system that is going to fill the students' heads with some specific knowledge and the dentist having to fill a tooth.

One agrees that the quality of the dentist is not defined by the very act of the filling of the tooth by the filler material, but by the relatively long and careful *preliminary treatment* of the hole to be filled. If this treatment is not done well, the filling falls out much before one has had the expected use of it. A qualified dentist has to well know what will be the future stresses on the filling.

Similarly to that, a teacher of a basic electrical engineering course has to well know what the future courses require from a basic course, in order to help consolidate the whole knowledge that the student will finally receive.

This background position clearly requires one to reconsider, from time to time, some circuit basics (which is done, however, very rarely), and the present attempt to extend one's view of the basic and very useful Thevenin's theorem, and also of some associated (which one might not expect) circuit nonlinearity, should be seen, in particular, in this context.

Let us start from the circuit of [1] and some terminology comments.

The circuit with the terminating dependent source

In [1], the case is considered where at the *output* of a one-port there is a *dependent source*. The controls of this dependent source are *inside* the one-port.

This source is *ideal*, that is, when its function is made zero by the controls, it becomes a short circuit *if it is a voltage source*, and a disconnection *if it is a current source*. In any case, it is not an "independent" (without any control) source for the circuit. Notice that "independent source" necessarily means "ideal source", but not conversely. However, if we speak only about independent sources, it would not be a mistake to (just) call each of them "ideal". Which type of source is under discussion will be clear from the context, in each case; just remember from where the possible controls are coming.

For certainty, we shall focus on the case when this source is a *voltage* source, and speak about Thevenin's (originally, *Helmholtz's*) version of the equivalent 1-port. Norton's version can be then obtained (see [1]) by duality. Figure 1 schematically shows the 1-port of [1] which interests us:

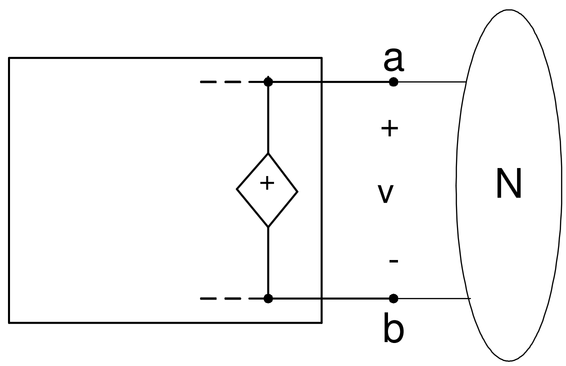

Fig. 1: The 1-port (including, in particular, also some *independent* sources) to the left of **a-b** should have, when detached, a unique solution ([1] for details). The controls of the dependent voltage source terminating the 1-port are solely internal, belonging to the 1-port. We consider the voltage $v = v_{\mathbf{ab}}$ on this source (the "port's voltage"). In [1], detailed examples are given for both linear and nonlinear circuit cases.

Figure 2 shows the equivalent for the circuit of Fig. 1, -- an ideal independent voltage source, relevant/valid, however, only when some limitations on the internal functional dependencies in the original circuit are provided. These limitations are very important.

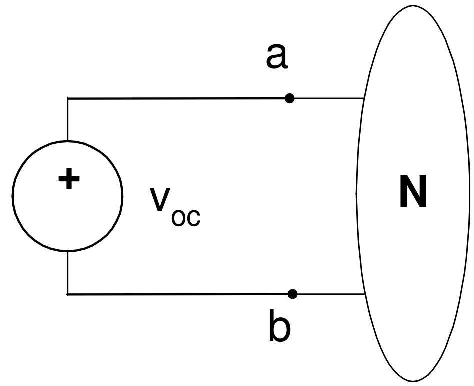

Fig. 2: The equivalent for the circuit in Fig.1, obtained with some limitations detailed in the main text. For the initially given linear circuit, this equivalent can be seen as "just" a case of Thevenin's theorem for $R_{Th} = 0$, but since the equivalent does not include any linear element, i.e. there is no evidence of circuit linearity, this equivalent can also represent a 1-port having nonlinear internal dependencies.

Considering that an ideal voltage source can conduct any current, i.e. that the *current* of the source depends on the load, it is shown in [1] that for the circuit of Fig. 1 there are two (and only two) possibilities:

A. If there is no feedback-influence of the *current* of the terminating voltage source on its controls (i.e. when the voltage on this source cannot be influenced by load's current), then with respect to the load or any other externally connected circuitry, the whole one-port appears to be an ideal *independent* voltage source.

B. If there *is* such an internal (unwanted in [1]) current feedback, acting as a "Trojan Horse", i.e. letting the load's current *internally* influence the 1-port (the voltage controls are our concern, of course), then the 1-port does *not* appear as an independent voltage source, i.e. $v = v_{\mathbf{ab}}$ depends on the parameters of the load or any other externally connected circuitry.

The proof for the case "A.", given in [1], is based on the fact that v can be determined without the nodal equation for the node **a** (see Fig.1), and thus no *independent* equation including the load's current arises. This proof assumes only existence of some functional relations, not their linearity.

That the original circuit need not be linear is also clear from the fact that the final equivalent scheme of the independent ideal source (Fig.2) does not include any linear element. This caused [1] to suggest (suppose) that an ideal source of a chaotic process can thus be obtained.

The lack in [1] of any systematic detailing of case "B." (just an example contrasting case "A.", is given there) is worth completing by including the results of [1] in a wider than usual frame for Thevenin's theorem. We think that this frame should be considered at the early stage of circuit theory studies, in particular because it naturally touches some important nonlinear features of the 1-ports.

The extended classification of the equivalents

In the following classification ***1-4*** of the equivalent circuits, both the theorems of [1] and the classical Thevenin's/Norton's theorem are used.

The first two (mutually dual) items specifically belong to the 1-port's topology of [1], and linearity of the circuit is not required. In some sense, this is a "pre-Thevenin's" case, letting the beginner better see what it may be "*an ideal source*".

Contrary to that, items ***3*** and ***4*** relate to both this topology and that (more usual) without the terminating dependent source. An important, far-reaching interpretation of the 1-port as a nonlinear resistor is introduced in item ***4***.

1. If the 1-port is terminated by a dependent (parallel) *voltage* source, and there is no internal *current* feedback from the source to its controls, then the whole 1-port presents an ideal (independent) voltage source with respect to its load or any other external circuitry connected to the port.

2. If the 1-port is terminated by a dependent (series) *current* source, and there is no internal *voltage* feedback from the source, influencing its controls, then the whole 1-port presents an ideal (independent) current source with respect to its load or any other external circuitry connected to the port.

3. If the circuit topology is not as in the above cases, or if it is such, but the formulated limitation on the internal feedback relations is violated, then if all the internal functional dependencies (including elements' characteristics and any control) are *linear*, we have (e.g., [2], or [3], for the nice standard proof), for any topology, for the current i, *entering* the port (thus, in (1), +, and not $-$) that:

$$v(i) = v_{Th} + R_{Th} i \,, \tag{1}$$

or, in Norton's dual form,

$$i(v) = -i_N + G_N v \,, \tag{1a}$$

where

$$G_N = \frac{1}{R_{Th}}, \quad \text{and} \quad i_N = G_N v_{Th} \,. \tag{2}$$

Observe that for the topology of Fig.1, the determination of R_{Th} has to be done, using a *current* test-source, while for the dual 1-port with the terminating *current* dependent source [1], it has to be done using a *voltage* test-source. (Indeed, ideal sources of the same kind can not be loaded on each other; this would be equivalent to short-circuiting a voltage source, -- consider thus Fig.1, -- or disconnecting a current source from any load.) A test-source is necessary [3] for any circuit including dependent source(s), but the present situation is not that of regular topology, when the test-source can always be either a voltage or a current one.

It is also important to recall that v_{Th} is nonzero *if and only if* the 1-port includes some independent sources.

Of course, the case of the linear circuit can be similarly formulated in terms of impedances instead of resistors.

4. To start this item, let us observe that for v_{Th} nonzero, (1) is a *nonlinear* dependence. More precisely, it is *affine*, i.e. without the direct proportionality required by system theory for "linearity" (it is not as in analytical geometry) of the *v-i* relation.

The usual case of the linear 1-port can thus be naturally included in the much more general case where the circuit's elements and controls are nonlinear, when we have, for the whole 1-port seen from outside, some nonlinear *v-i* dependence

$$v = v(i) \tag{3}$$

found from the circuit (Kirchhoff's) equations, without any *a priori* seen simplifying possibilities.

Giving the "uncomfortable" nonlinear case especial importance, we thus see (1) as a particular case of (3), i.e. as some "zero nonlinearity", which means linearity of the circuit *defined without the independent sources*. The meaning of the latter stress will be immediately explained.

An electrical circuit can be a jewel seen either as linear or nonlinear, depending on the "angle of vision"

Remarkably, in *each* of the classified cases, a *nonlinear* resistor $v(i)$ is observed from outside. This shows, -- and with compliments to [2] and [5,6], -- the exceptional importance of the use of the notion "nonlinear resistor" during the early studies.

Though in connecting a linear passive circuit to an independent source we obtain a linear response, when examining *then* this circuit from some output, we not only face the understandable fact that, as a rule, a circuit is seen differently from different inputs (test entries); the more interesting fact is that the initially given circuit is drastically changed by the source which is now *included in it*.

First of all, it is very important here that any ideal source, taken by itself, is a strongly nonlinear element. For instance, a battery *whose voltage is independent of its current* is a "saturated" element or a "voltage hardlimiter"; i.e. it obviously is a strongly nonlinear element. Thus, when *belonging* to a circuit, and *not* being its input (!) the battery makes this circuit nonlinear.

In order to make this role of the *choice of input* absolutely clear, let us consider Figs.3(a,b,c) illustrating the simple principle that for considering a circuit as a linear one, one has to carefully consider what are its inputs

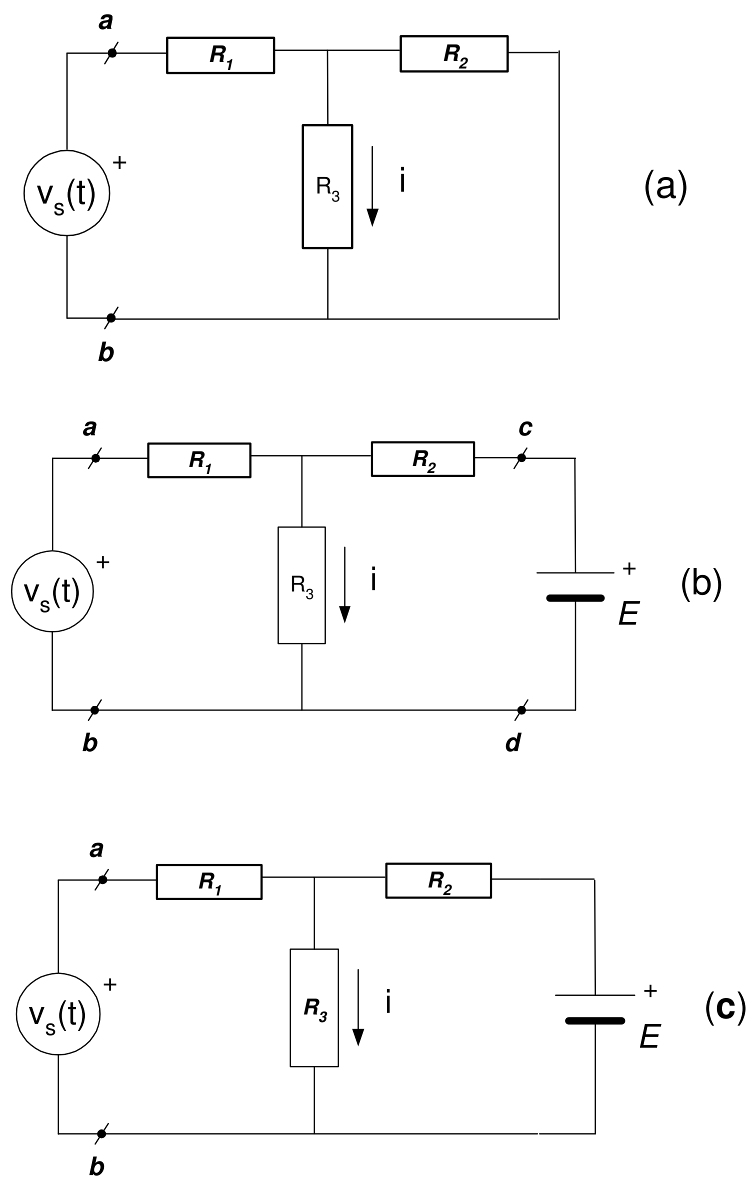

Fig. 3 (**a**,**b**,**c**): Illustration of the fact that whether or not a circuit is linear depends on *definitions (choice) of its inputs.* All the resistors are *LTI*; current i is the output variable.
(**a**) *Linear* circuit with single input at (a-b), v_s;
(**b**) *Linear* circuit with *two* scalar inputs, at (a-b) and (c-d), v_s and E;
(**c**) *Nonlinear* circuit (thus seen from its single input (a-b)) including the battery that now is not any input, just a (strongly nonlinear) *circuit element*. (Think about the battery in terms of the basic *substitution theorem* [2]!)

In case **a**, the input is v_s, and the response of the circuit is linear,

$$i = Kv_S \tag{4}$$

with a nonzero constant K.

In case **b**, the battery E is added *as the second scalar input*, and the response is:

$$i = K_1 v_s + K_2 E, \tag{5}$$

with some nonzero constants K_1 and K_2, which is a linear form of v_s and E. In matrix notation, when the single *vector* input, $[v_s, E]^T$, is defined, (5) becomes similar to (4):

$$i = \left[K_1, K_2\right] \begin{bmatrix} v_s \\ E \end{bmatrix}. \tag{5a}$$

The latter form (motivated by the theory of state equations; undoubtedly, the best tool for treatment of linear systems) is preferable; it is most suitable to speak about *one input* for performing the test of linearity.

Thus, the present circuit is also linear.

In case **c**, we can start from (5) as an auxiliary equation, but then have to see it in the sense of (1), i.e. as

$$i = K_1 v_s + K_3 \tag{6}$$

with the single defined input v_s and a new *fixed* constant $K_3 \neq 0$. Obviously, it is a nonlinear relation, which neither of the scaling and the additive tests of linearity pass.

Note that one *cannot just* say here that $K_3 = K_2 E$, without adding that the parameter E is now "dead", fixed; i.e. it already does not belong to a *set* of the parameters and functions from which we are picking an input function, freely (e.g. continuously) *changing it for performing the test of linearity*, say, arbitrarily changing the scaling: $f \to kf$, $\exists k$.

This "fixation forever", associated with the definition of the source as an internal circuit element, or with approaching the circuit with some internal sources *as just a 1-port*, creates the nonlinearity.

Thus, the activeness of a circuit can be expressed in a nonlinearity, and, more generally, we touch at this point the important problem [7] of the definition of a system which would be correct from the system-theoretic outlook, i.e. from *both* the general mathematics and the concrete engineering (where the sense of the inputs arises) points of view.

In [7], a nonlinear and a linear *time-variant* versions of a system are thus compared, and it is shown that incorrect interpretation of a given (fixed) function as an input can lead to a wrong definition of a system as linear or nonlinear. The present example is simpler and it should be easier to start the point with it.

We have thus extended not only the frame of Thevenin's theorem, but also the related possibility to see a circuit as a nonlinear one, considering that it is desirable not to separate these things. Returning to the observation in the previous section that for a test, made using an active external circuit, *nonlinear* 1-ports are obtained in *all four cases* of the extended classification, we can suggest considering the following statement:

If one wishes to compose, using linear elements and ideal sources, circuit that would be seen as linear, independently of the choice of the output variable and independently of the choice of the entrance (input), using which the test of linearity is *performed; one must define the circuit so that all the sources that finally appear as independent remain outside the circuit, i.e. are its defined inputs, while the sources that finally appear to be dependent sources can all be inside the defined circuit, together with the linear elements.*

That is, the circuit's *interior* and *input* (generally, in the vector sense) are basic concepts for the definition of a circuit, and by themselves must be defined very clearly.

Comments

1. Generalization of the argument of passing on to nonlinearity (as from (5) to (6)) on *N-ports* is immediate. Just delete the title "input", -- *that is, the rights to be changed for performing the test of the linearity* -- *from a certain independent input*, *or fix this particular applied function*, and a linear *N*-port becomes a nonlinear (*N*-1)-port, because the linear input-output relation can become the nonlinear affine dependence, just as (6).

2. Generalization of such circuits as in Fig.3(a,b,c) to the circuits including inductors and capacitors is also easily done, using Laplace transform, with impedances instead of resistors. Then, in the *time* domain, the above linear *algebraic* relations become linear *convolution* relations. The argument of affine nonlinearity remains in, basically, the same terms.

3. When interpreting a battery as a strongly nonlinear element that makes, when included in a circuit, this circuit nonlinear, we use, in fact, the basic *substitution theorem*. However, contrary to that, for instance, we replace, in the proof [3,2] of the usual Thevenin's theorem, the external circuit connected to the 1-port by an ideal voltage source; here we replace a source by a circuit (an element). This "converse" use of the mathematical equivalence of a source and an element, stated by the substitution theorem, is worth a special stress in the basic theory. Compare with [2] where only "direct" examples are found.

4. The fact that the interpretation of such a 1-port as a nonlinear resistor is completely unmentioned in the classical view of Thevenin's/Norton's theorem becomes today a pedagogical omission. It seems, for instance, that it should be useful for the theory of electronics amplifiers, studied later. The very detailed consideration of resistive circuits in [2] (see also [5,6]) is good background reading and can help here, but this part of [2] is too "heavy" and far from the immediately found applications for many students and teachers, and thus the topic of nonlinearity is often completely omitted, e.g. in [3], to point at one among many such known standard textbooks. Hopefully, the present simple arguments, connecting the topic of nonlinearity with the popular Thevenin's theorem, can cause more interest to arise in nonlinear resistive circuits.

Figure 4 shows the connections between the topics touched.

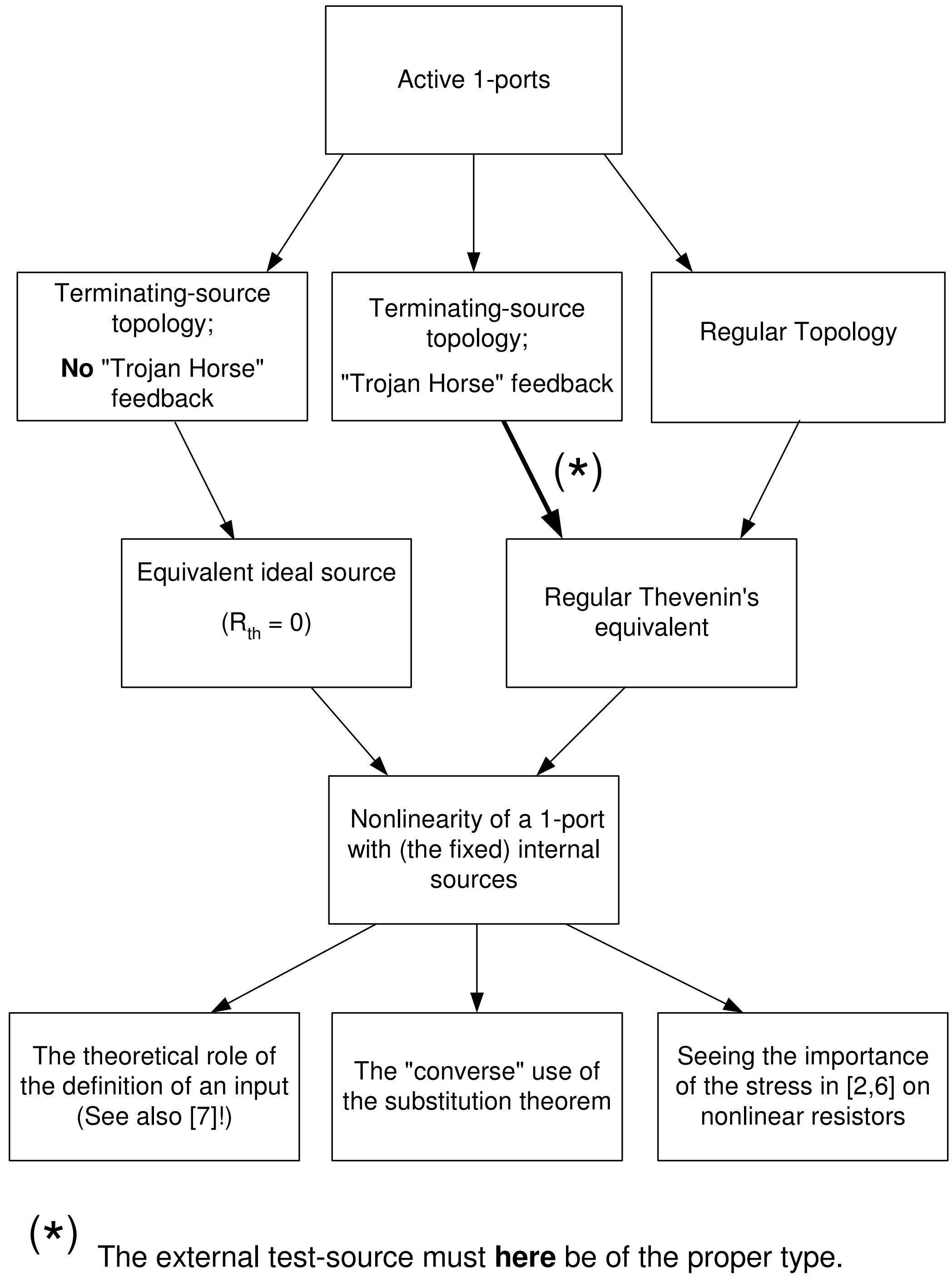

Fig.4: The logical scheme of the argument.

A historical-view remark about Thevenin theorem

The already-mentioned distinction between the system theory outlook and the purely mathematical outlook becomes clearer if we consider the fact that the theory of linear *static and dynamic* systems and equations started in mechanics, and not in electrical engineering. Thus, why did Helmholtz, who was able (!) to formulate (in 1847) the energy conservation law for both mechanical and electrical systems, formulate (in 1853) the theorem only for electrical circuits? In other words:

Why didn't Thevenin's theorem appear first in the theory of mechanical systems (with a mechanical force instead of the battery, etc., which, from the purely analytical outlook, is just the same), and then come to the theory of electrical systems which appeared later?

This happened so because *in electrical engineering* the concept of *input* became more flexible, allowing one to see the system from its different "sides", which basically means the wider range of possibilities to actually connect the system to a measuring device, or a load, etc.. Only in this way, Thevenin's circuit obtained its general importance that, in particular, involves, as we argue here, the aspect of the realization of a nonlinear resistor.

The wide use of the different possible inputs in electrical engineering (and then in the art of electronics) explains the very serious attention to nonlinear resistors in [2], compared to that in the courses of mechanics where nonlinear friction elements are considered but are not realized by means of an initially linear system made nonlinear by some independent forces included in it.

Open problems for *teaching* (*a-d*) and *research* (*e-g*)

a. Include the cases of [1] in the regular teaching of circuit equivalents;

b. Consider the application of the "extended frame" to the theory of active 2-port circuits, which is interesting in itself and is a good modern introduction to the theory of amplifiers and active filters.

c. Find application of the "extended frame" in the detailed theory of amplifiers where the circuits with terminating dependent sources can arrive. (Remember that in cases "1." and "2." of the classification, nonlinear internal functional relations are permitted, i.e. such a circuit cannot, generally, be given in terms of the impedances.)

d. Discuss with your students an electrical circuit as "linear-nonlinear", dependently on the definition of the inputs. Let the students see that the classical Thevenin's (Northon's) equivalent for a circuit, including voltage (current) sources and linear resistors, is tested as a nonlinear resistor, simply because we approach the circuit as a 1-port. Consider [8] in order to see that affine nonlinearity can be a special case of the more usual "zero-crossing nonlinearity". Find interesting examples and prepare lesson: "Nonlinearity of active circuits".

e. Create a circuit (1-port) model in which the unwanted in [1] "Trojan Horse" feedback appears from time to time, i.e. the circuit equivalent is alternating between the ideal-source form and the usual Thevenin's form.

f. Combine, in the nonlinear case, a chaotic oscillator with the creation of the equivalent source (items 1 or 2 of the classification), and thus obtain an ideal source, either voltage or current, of the chaotic process.

g. Give an analysis of possible non-uniqueness of the solutions in all of the cases here where nonlinearity is permitted.

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